# Expressiveness and Complexity of Similarity Quantifiers

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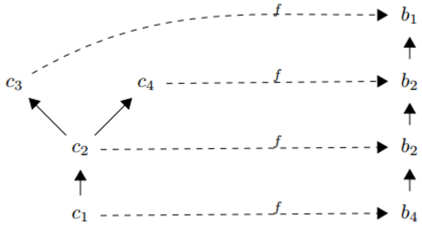
# Motivation

## [Barwise, 1979]

*The richer the country the more powerful are some of its officials.*




## Thesis ([Barwise, 1979])

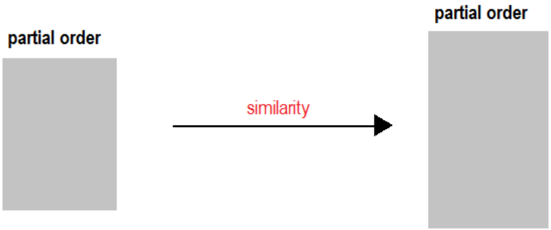
*Logical form of such sentences is best captured in SO.*

## Theorem ([Barwise, 1979])

*Homomorphism is not FO-expressible over arbitrary models.*

# Overview of the Present Work

1. other notions of **similarity** between partial orders




2. restr. to **finite models** [Westerståhl, 1984, Szymanik, 2016]
3. generalized quantifiers theory → **similarity quantifiers**
4. **expressiveness** → extension to Barwise's results
5. **complexity** of s.q. → not explored previously
   [Mostowski and Wojtyniak, 2004, Szymanik, 2010]

# Similarity Quantifiers

- Relational vocabulary $\sigma = (A, <_A, B, <_B)$
- $(A, <_A)$, $(B, <_B)$ - finite strict partial orders

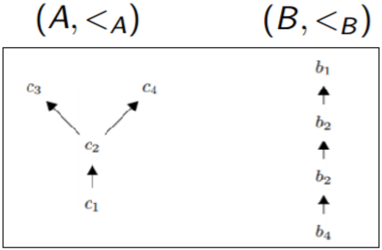
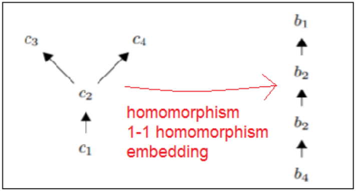



Figure 1: Double partial order

# Similarity Quantifiers

$(A, <_A)$ $(B, <_B)$




**Homomorphism,** $\mathcal{H}$

$$\exists f : A \rightarrow B \, \forall x, y \in A [(x <_A y \Rightarrow f(x) <_B f(y))]$$

**1-1 Homomorphism,** $\mathcal{H}^{1-1}$

$$\exists f : A \stackrel{1-1}{\rightarrow} B \, \forall x, y \in A ((x <_A y \Rightarrow f(x) <_B f(y))$$

**Embedding,** $\mathcal{E}$

$$\exists f : A \stackrel{1-1}{\rightarrow} B \, \forall x, y \in A ((x <_A y \Leftrightarrow f(x) <_B f(y))$$

# Similarity Quantifiers

- Relational vocabulary $\sigma_R = (A, <_A, B, <_B, R)$
- coupling relation $R \subseteq A \times B$

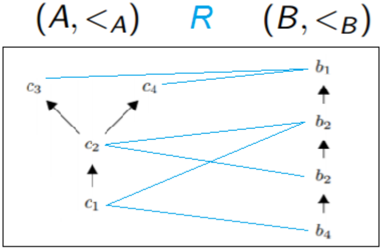



Figure 2: Coupled partial order

$$R_a := \{b \in B : R(a, b)\}$$

- requirement for similarity function $f$: $\forall\, a \in A\; f(a) \in R_a$.

# Similarity Quantifiers (Restricted Versions)

**Restricted Homomorphism,** $\mathcal{H}_r$

$$\exists f : A \to B \, \forall x, y \in A [\, R(x, f(x)) \; \wedge \; (x <_A y \Rightarrow f(x) <_B f(y))]$$

**Restricted 1-1 Homomorphism,** $\mathcal{H}_r^{1-1}$

$$\exists f : A \overset{1-1}{\to} B \, \forall x, y \in A (\, R(x, f(x)) \; \wedge \; (x <_A y \Rightarrow f(x) <_B f(y))$$

**Restricted Embedding,** $\mathcal{E}_r$

$$\exists f : A \overset{1-1}{\to} B \, \forall x, y \in A (\, R(x, f(x)) \; \wedge \; (x <_A y \Leftrightarrow f(x) <_B f(y))$$

# Similarity Quantifiers

### Disjointness of $\{R_a\}_{a \in A}$

$R_a \cap R_b = \emptyset$, for every $a, b \in A$ such that $a \neq b$ (FO-sentence)

$\Downarrow$ add this condition $\Downarrow$

$$\mathcal{H}_{rd}, \mathcal{H}_{rd}^{1-1}, \mathcal{E}_{rd}$$

# Indefinability of Similarity Quantifiers in Finite Models

Barwise [Barwise, 1979] proved that that homomorphism between partial orders is not *FO*-expressible over arbitrary models.
What we prove is

**Theorem**

*$\mathcal{H}, \mathcal{H}^{1-1}, \mathcal{E}$ are not FO-definable over double partial orders.*

**Proof.**

If $\mathcal{H}, \mathcal{H}^{1-1}$ or $\mathcal{E}$ is *FO*-expressible over double partial orders then parity is expressible over linear orders, a contradiction. □

# Indefinability

**Theorem**

*$\mathcal{H}_r$, $\mathcal{H}_r^{1\text{–}1}$, $\mathcal{E}_r$, $\mathcal{H}_{rd}$, $\mathcal{E}_{rd}$ are not FO-definable over coupled partial orders.*

The proof is by Hanf-locality argument.

# Complexity

- What is the computational complexity of similarity quantifiers?
- relevance of complexity measures for cognition [Szymanik, 2016]
- how hard it is to use such constructions?

verify them against finite situations

Analogous questions explored wrt different constructions [Mostowski and Wojtyniak, 2004, Sevenster, 2006a, Szymanik, 2010]

# Complexity

### Definition

Let $\mathcal{A} = (A, <_A)$ be a finite strict partial order. The height of $\mathcal{A}$, denoted by $h(\mathcal{A})$, is the number of vertices in the longest chain in $\mathcal{A}$.

### Lemma

*$h$ is in $P$.*

### Proof.

The argument uses the Sedgewick trick [Sedgewick and Wayne, 2011] for finding the longest paths in a directed graph with the help of the Ford-Bellman algorithm □

# Complexity

**Lemma**

*Let $\mathcal{A}$, $\mathcal{B}$ be strict posets. There is a homomorphism from $\mathcal{A}$ to $\mathcal{B}$ iff $h(\mathcal{A}) \leq h(\mathcal{B})$.*

**Theorem**

*$\mathcal{H}$ is in $P$.*

**Proof.**

Corollary from the Lemmata. $\square$

# Complexity

**Theorem**

*$\mathcal{H}_r$, $\mathcal{H}_{rd}$, $\mathcal{H}^{1-1}$, $\mathcal{H}_r^{1-1}$, $\mathcal{E}$, $\mathcal{E}_r$, $\mathcal{E}_{rd}$ are NP-complete.*

**Proof.**

Each of those quantifiers is definable by existential *SO*-sentences. Hence, by the Fagin theorem [Fagin, 1974a], they are in *NP*. To prove that they are *NP*-hard, we show 3SAT is polynomially reducible to each of those quantifiers.

□

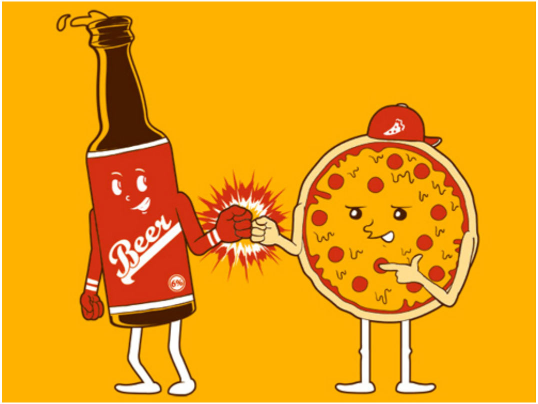
# Conclusions

- in view of Ristad's thesis [Ristad, 1993, Mostowski and Szymanik, 2012], similarity quantifiers are among the strongest properties expressible in everyday language
- similarity quantifiers (except for raw homomorphism) are thus among the hardest everyday language concepts (see, e.g., [Mostowski and Wojtyniak, 2004, Szymanik, 2010])
- unrestricted homomorphism is practically verifiable
- further linguistic implications are developed in [Kalociński and Godziszewski, 2018]

## General Question

What are the properties of logics with similarity quantifiers?

Thank you for your attention!



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