# Degree spectra of computable functions on $(\omega, <)$

D. Kalocinski[a][1]

joint work with
N. Bazhenov[b] M. Wrocławski[c]

[a]Institute of Computer Science, Polish Academy of Sciences, Poland

[b]Sobolev Insitute of Mathematics, Novosibirsk, Russia

[c]Faculty of Philosophy, University of Warsaw, Poland

Mal'tsev Meeting, 22 Sept 2021

[1]Funded by grant no. 2018/31/B/HS1/04018 of the National Science Centre Poland.

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## Introduction

# Motivations

- Core theme in *computable structure theory* —characterise the range of complexity of a relation across all effective copies of the underlying structure.

A structure $\mathcal{A}$ is computable if its domain and basic relations are uniformly computable.

Definition (Ash and Nerode 1981). The degree spectrum of a computable $R \subseteq A^n$ on computable $\mathcal{A}$, denoted $DgSp_{\mathcal{A}}(R)$, is the set of all Turing degrees of images of $R$ under all isomorphisms $f : \mathcal{A} \to \mathcal{B}$ with $\mathcal{B}$ computable.
If $DgSp_{\mathcal{A}}(R) = \{0\}$, then we say that $R$ is intrinsically computable.

- Wright (2018) asks for a more comprehensive characterisation of degree spectra of computable $n$-ary relations on $(\omega, <)$.
- computable copies of $\omega \leftrightarrow$ notations for naturals (Shapiro, 1982)
- spectra of $R \sim$ measure of complexity of $R$ in notations with computable ordering

PROBLEM: What are the degree spectra of unary total recursive functions on $(\omega, <)$?

Henceforth, $f$ is a (graph of a) total unary recursive function and $DgSp$ means $DgSp_{(\omega,<)}$.

**Theorem (Moses 1986).** $DgSp(f) = \{0\}$ iff $f$ is almost identity or almost constant.

**Theorem (Wright 2018).** For computable $R \subseteq \omega$, $DgSp(R)$ is $\{0\}$ or $\Delta_2$ degrees.

**Theorem (Theorem 4.7. in Wright 2018, extending Downey et al. 2009).**
For computable $R \subseteq \omega^n$, either $DgSp(R) = \{0\}$ or it contains all c.e. degrees.

$f$ total computable (but not intrinsically) $\rightarrow$ c.e. degrees $\subseteq DegSp(f) \subseteq \Delta_2$ degrees.

# Wright's question

**Theorem (Theorem 4.7. in Wright 2018, extending Downey et al. 2009).**
For computable $R \subseteq \omega^n$, either $DgSp(R) = \{0\}$ or it contains all c.e. degrees.

*Question 6.2. How can Theorem 4.7 be generalized?*

*Theorem 4.7 tells us that the degree spectrum of a computable n-ary relation on $(\omega, <)$ is computable or contains all c.e. degrees, but as we've seen, this does not fully characterize the possible degree spectra (as there are relations whose spectrum is exactly the c.e. degrees, and relations whose spectrum is all $\Delta_2$ degrees). Are there computable relations on $(\omega, <)$ with other degree spectra, and more generally, what are the possible degree spectra?*

## Results

# Functions with finite range

Using methods similar to Wright's, it is possible to show that a stronger theorem holds.

**Theorem 1**

If $f : \omega \to \omega$ has a finite range, then its degree spectrum is trivial or $\Delta_2$.

The above fails for some *functions with infinite range*.

**Example**

The degree spectrum of the successor consists of precisely c.e. degrees.

# Block functions

**Definition.** Total $f$ is a *block function* if for all $a \in \omega$, there is a finite interval $I$ of $(\omega, <)$ s.t.:

- $a \in I$;
- $I$ is closed under $f$ (i.e, for all $x \in I$, $f(x) \in I$);
- $I$ is closed under $f^{-1}$ (i.e., for all $x \in I$, $f^{-1}(x) \subseteq I$).

For the $\subseteq$-least such interval $I$ the structure $(I, <, f \restriction I)$ is an *$f$-block* (of the element $a$).
If $(I, <, f \restriction I)$ is an $f$-block, we refer to its isomorphism type as an $f$-type (or a type).

1. computable block function $f$ has 1-1 computable enumeration of its types
2. $f$ represented by $\alpha_f \in [0, N]^\omega$, where $[0, N]$ is the domain of the enumeration

**Example.** $f(n) = 2 \cdot \lfloor \frac{n}{2} \rfloor$ is a block function.

# Theorem on block functions with finitely many types

**Theorem.** If $f$ is a block function with finitely many types, then $DgSp(f)$ is $\{0\}$ or $\Delta_2$ degrees.

$N$ types, combinatorics on $\alpha \in [1, N]^\omega$ and constructions for the following cases:

1. There are two different strings $\sigma, \tau \in [1, N]^*$ such that:
   - the lengths of $\sigma$ and $\tau$ are the same;
   - $\tau$ can be obtained via a permutation of $\sigma$
   - both $\sigma$ and $\tau$ occur infinitely often in $\alpha_f$.
2. There is only one type occuring infinitely often in $\alpha_f$.
3. Neither of the previous two cases holds.

**Example**

There is a block function with inf. many types with spectrum equal to the c.e. degrees.

# Quasi-block functions

**Definition.** $f$ is a quasi-block function if there are arbitrarily long finite initial segments of $\omega$ closed under $f$ (proper quasi-block function = quasi-block function but not block function).

**Example.** Euler's $\varphi$, $n \mapsto$ number of divisors of $n$.

**Theorem.** Spectrum of unary total computable non-quasi-block $f$ is equal to the c.e. degrees.

**Theorem.** Spectrum of a proper quasi-block $f$ having a computable non-decreasing lower bound $g$ such that $\lim_n g(n) = \infty$ is equal to the c.e. degrees.

**Theorem.** There exists a computable quasi-block $f$ with a non-decreasing lower bound diverging to $\infty$ but with no such computable bound. Moreover, there is such $f$ whose spectrum is equal to the c.e. degrees.

# Unusual degree spectrum

Recall the following:

**Theorem (Moses 1986).** $DgSp(f) = \{0\}$ iff $f$ is almost identity or almost constant.

**Theorem (Theorem 4.7. in Wright 2018, extending Downey et al. 2009).**
For computable $R \subseteq \omega^n$, either $DgSp(R) = \{0\}$ or it contains all c.e. degrees.

We prove:

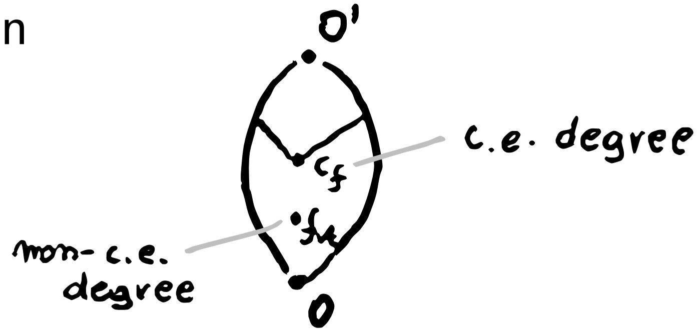
**Theorem (on unusual degree spectrum).** There exists a total computable block function $f$ whose spectrum is different from $\{0\}$, the set of all c.e. degrees and the set of all $\Delta_2$ degrees.

The counting function for a block function $f$ is defined by $c_f(n) = \#\{i : \alpha_f(i) = n\}$.

**($\star$)** Let $f$ be a computable block function with infinitely many pairwise non-embeddable types, each occurring finitely often. Then deg($c_f$) is c.e. and $f_{\mathcal{A}} \geq_T c_f$ implies that $deg(f_{\mathcal{A}})$ is c.e.

We construct a computable block function $f$ and a computable copy $\mathcal{A}$ of $(\omega, <)$ such that:

- $f$ has infinitely many types each occurring finitely often
- $c_f <_T 0'$
- $f_{\mathcal{A}}$ is of non-c.e. degree.




Combining this with **($\star$)** and

**Theorem (Cooper, Lempp and Watson, Cooper et al. (1989)).** Given c.e. sets $U <_T V$ there is a set $C$ of properly d.c.e. degree such that $U <_T C <_T V$.

finishes the proof.

# Open Questions

# Problems

Characterise degree spectra on $(\omega, <)$ of total computable functions with $\infty$-many block types.

Theorem. Let $f$ be a recursive block function having infinitely many block types with almost every type occurring infinitely often. Then $DgSp(f)$ contains a proper 2-c.e. degree.

## Question

Are there infinitely many possible degree spectra on $(\omega, <)$ for unary total computable functions?

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