Learnability Thesis Does Not Entail Church's Thesis*

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Abstract

We consider the notion of intuitive learnability and its relation to intuitive computability. We briefly discuss the Church's Thesis. We formulate the Learnability Thesis. Further we analyse the proof of the Church's Thesis presented by M. Mostowski. We indicate which assumptions of the Mostowski's argument implicitly include that the Church's Thesis holds. The impossibility of this kind of argument is strengthened by showing that the Learnability Thesis does not imply the Church's Thesis. Specifically, we show a *natural* interpretation of intuitive computability under which intuitively learnable sets are exactly algorithmically learnable but intuitively computable sets form a proper superset of recursive sets.

Keywords: computability, algorithmic learnability, potential infinity, FM-representability, low sets, Learnability Thesis, Church's Thesis

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1 Intuitive Computability and the Church's Thesis

Before the emergence of computability theory as a branch of modern logic, many algorithms had been known. Historically, the first non-trivial algorithm, the Euclidean algorithm, dates back to circa 300 BC when the Greek mathematician, Euclid of Alexandria, formulated his method for calculating the greatest common divisor. In 1900, shortly before the appearance of the first mathematical models of computation, Hilbert formulated his tenth problem of finding an algorithm for deciding whether a given equation is solvable in integers. These and many other historical examples convince us that even before the era of computability we had some intuitive notion of algorithm, precise enough to be incorporated by science. The era of the computability theory started in the 1930s and was marked with the appearance of the first mathematical models of computation [1], [4], [9], [13]. Almost immediately the following question arose: are the notions of intuitive computability and, for example, λ-definability or, what comes to the same thing, Turing-computability, equivalent? In other words, is the class of intuitively computable sets equal to the class of recursive sets? The affirmative answer to this question is known as the Church's Thesis and was first formulated in [1], [13].[1] The Church's Thesis, if not treated as definition, and we actually do not treat it as such,[2] is a statement about the equality of two classes of objects. From now on, by $\mathcal{IC}$ we mean a subset of $\mathcal{P}(\omega)$, consisting of intuitively computable sets of natural numbers. The class of recursive sets is known to be Δ^0_1 in arithmetical hierarchy.[3] Having this notation, the Church's Thesis presents shortly as follows:

Thesis 1 (**Church's Thesis**). $\mathcal{IC} = \Delta^0_1$.

The inclusion $\Delta^0_1 \subseteq \mathcal{IC}$ is generally accepted as a rule. The whole mystery lies in $\mathcal{IC} \subseteq \Delta^0_1$. $\mathcal{IC}$ is not fully understood. We have some intuitions based on practice in devising intuitive algorithms and writing computer programs. Our intuitions are strengthened by deep insights of computability theory.

[1]Another formulation of the Church's Thesis, in terms of functions, states that the class of intuitively computable functions is identical with the class of partial recursive functions. We restrict ourselves to the first formulation.

[2]Observe that treating $\mathcal{IC} = \Delta^0_1$ as a definition of $\mathcal{IC}$ strips away the whole problem, since then, $\mathcal{IC} \subseteq \Delta^0_1$ holds.

[3]For a detailed exposition of arithmetical hierarchy see, for example, [11].

However, it is still possible, though unlikely, that $\Delta_1^0 \neq \mathcal{IC}$. This possibility is essentially used in the proof of the main theorem (Theorem 6) which is based on the notion of intuitive learnability.

2 Intuitive Learnability and the Learnability Thesis

The notion of intuitive learnability is based entirely on the notion of intuitive computability. Given the intuitive notion of an algorithm, one can define the notion of intuitive learnability as follows:

Definition 1 (**Intuitive Learnability**). *A decision problem is intuitively learnable if there is an intuitive algorithm that for each example of the problem runs* ad infinitum *and produces a finite sequence of yeses and nos such that the last answer in the sequence is correct.*

The origins of the notion of intuitive learnability can be traced back to the same Euclid of Alexandria that is known as the author of the first non-trivial algorithm. His *Elements* contains the first exposition of the axiomatic method. The search for a proof of a sentence in a given axiomatic system may be viewed as an example of an intuitive algorithm that generates a finite sequence of answers as to whether the input sentence is provable. At the beginning the negative answer is produced. Then the space of proofs is systematically explored. If the input sentence is provable, the exploration finishes once the proof is found, the positive answer is produced, the algorithm stops and the generated sequence of answers is "no", "yes", with the last answer being correct. If the input sentence is not provable, the exploration goes on forever, and the generated sequence is always "no". This intuitive algorithm shows that the set of theorems of a recursive set of axioms is intuitively learnable.

Modern science, dating back to 17th century, provides another example of intuitive learnability. Consider a simplified model of the activity of a modern scientist. The scientist proposes a system of hypotheses. The system is to describe the world correctly. Initially, the positive answer is produced, meaning that hypotheses are considered true. Then the scientist proceeds to testing. If hypotheses are correct, testing goes on forever and the generated sequence of answers is always "yes". If hypotheses are incorrect, some test fails, the negative answer is produced, the activity stops and the generated sequence of answers is "yes", "no", with the last answer being correct. This intuitive algorithm shows that the problem of whether a system of empirical hypotheses describes the world correctly is intuitively learnable.[4]

The axiomatic method and the scientific method had appeared long before 1960s when algorithmic learning theory was established. The emergence and endurance of these sophisticated learning techniques provide a rationale that we had some intuitive understanding of learnability in times preceding its mathematical models.

Mathematical notion of learnability is due to Gold [3] and Putnam [10]. Here is Putnam's definition of algorithmic learnability that accounts for a mathematical counterpart of an intuitive idea of a set "decidable" by a mind-changing procedure:

Definition 2 (**Algorithmic Learnability**). *Let $A \subseteq \omega$. A is algorithmically learnable if there is a total computable function $g : \omega^2 \to \{0, 1\}$ such that for all $x \in \omega$: $\lim_{t\to\infty} g(t, x) = 1 \Leftrightarrow x \in A$ and $\lim_{t\to\infty} g(t, x) = 0 \Leftrightarrow x \notin A$.*

Algorithmic learnability is equivalent with many natural notions. One of them is the notion of FM-representability proposed by Mostowski in [8]. His research was motivated by computational foundations of mathematics and the search for the semantics under which first-order sentences would be interpreted in potentially infinite domains. Potentially infinite domains are understood as growing sequences of finite models. We consider the latter to have purely relational vocabulary and initial segments of natural numbers as universes. Let $R \subseteq \omega^r$. Then by $R^{(n)}$ we denote $R \cap \{0, 1, \ldots, n\}^r$. For any model on natural numbers $\mathcal{A}$ over the signature $\sigma = (R_1, \ldots, R_k)$ we define the FM-domain of $\mathcal{A}$ as follows: $\mathrm{FM}(\mathcal{A}) = \{\mathcal{A}_n : n \in \omega\}$, where $\mathcal{A}_n = (\{0, 1, \ldots, n\}, R_1^{(n)}, \ldots, R_k^{(n)})$. By $\mathbb{N}$ we denote the standard model of arithmetic $(\omega, R_+, R_\times)$ of the vocabulary $\sigma = (R_+, R_\times)$, where instead of function symbols $+$, $\times$, we have corresponding relational symbols R_+, $R_\times$, interpreted in the same way as $+$, $\times$.

Definition 3 (**FM-representability**). *We say that the relation $R \subseteq \omega^r$ is FM-represented in $\mathrm{FM}(\mathcal{A})$ by a formula $\varphi(x_1, \ldots, x_r)$ if and only if for each*

[4]Our description is simplified. However, it seems, that it captures the main idea, that the system of empirical hypotheses cannot be conclusively justified but can be conclusively rejected (falsified).

$a_1, \ldots, a_r \in \omega$ both of the following conditions hold:

$$R(a_1, \ldots, a_r) \text{ if and only if } \exists m\, \forall k \geq m \;\; \mathcal{A}_k \models \varphi(a_1, \ldots, a_r) \qquad (1)$$

$$\neg R(a_1, \ldots, a_r) \text{ if and only if } \exists m\, \forall k \geq m \;\; \mathcal{A}_k \models \neg\varphi(a_1, \ldots, a_r) \qquad (2)$$

*We say that R is **FM-representable** in $\mathrm{FM}(\mathcal{A})$ if there is a formula φ such that it FM-represents R in $\mathrm{FM}(\mathcal{A})$. If a relation is FM-representable in $\mathrm{FM}(\mathbb{N})$ we say that it is **FM-representable**.*

FM-representability is a good model of the semantic meaningfulness of mathematical concepts that we learn. The simplest argument is that objects, concepts and phenomena that are in the scope of cognitive accessibility and computational tractability for a human mind are of a finite character. Even if it is actually infinite, we may experience only its finite parts - hence we assume that the only epistemically reasonable notion of infinity we may adopt is the notion of potential infinity, explicated within the framework of FM-domains.

Subsequent theorem is a collection of notions that turned out to be equivalent to algorithmic learnability.

Theorem 1 (**Limit Lemma**). *Let $R \subseteq \omega^r$. Then the following are equivalent:*[5]

1. *R is recursive with recursively enumerable oracle,*
2. $\deg(R) \leq 0'$,
3. *R is algorithmically learnable,*
4. *R is Δ^0_2,*
5. *R is FM-representable.*

Algorithmic learnability and equivalent notions given in the Limit Lemma are of mathematical nature. However, as we showed in the Definition 1, the notion of learnability has also a very clear intuitive content that may be formulated using the notion of intuitive computability. Therefore we actually

[5] The equivalence between 1, 2 and 4 is due to Shoenfield [12]. The equivalence between 3 and 4 is due to Gold [3] and Putnam [10]. The equivalence between 1-4 and 5 is due to Mostowski [5], [8] and is called the FM-representability theorem.

have two notions of learnability: the intuitive one, given in the Definition 1, and the mathematical one, given by any of equivalent statements in the Limit Lemma. And just as in the case of the notions of intuitive and, for example, Turing-computability, we face the question of whether the notion of intuitive learnability is equivalent to the notion of algorithmic learnability. In other words, is the class of intuitively learnable sets equal to the class of algorithmically learnable sets? We put forward a claim, under the name of Learnability Thesis, that intuitive learnability is equivalent to algorithmic learnability. From now on, by $\mathcal{IL}$ we mean a subset of $\mathcal{P}(\omega)$, consisting of intuitively learnable sets of natural numbers. The class of algorithmically learnable sets is, by the Limit Lemma, Δ^0_2. Having this notation, the Learnability Thesis presents shortly as follows:

Thesis 2 (**Learnability Thesis**). $\mathcal{IL} = \Delta^0_2$.

At this point, a natural question to ask is: why should we accept this claim? It is not our main purpose to argue in favour of the Learnability Thesis (we need it in our argumentation for the impossibility of the specific kind of proof of the Church's Thesis). Nevertheless, as the Limit Lemma indicates, algorithmically learnable sets form a very natural class of objects. So far, the class has been discovered by researchers from three different domains: computability theory (Shoenfield), artificial intelligence (Gold), logic and philosophy (Putnam, Mostowski). Moreover, it is easy to see, that $\Delta^0_2 \subseteq \mathcal{IL}$ – the argument goes analogously to the one that shows $\Delta^0_1 \subseteq \mathcal{IC}$. The tricky part is $\mathcal{IL} \subseteq \Delta^0_2$. However, assuming the Church's Thesis, the argument trivialises (we provide it only for illustrative purposes).

Proposition 1. *The Church's Thesis entails the Learnability Thesis.*

Proof. Assume the Church's Thesis.

($\Delta^0_2 \subseteq \mathcal{IL}$) Let $A \in \Delta^0_2$. Let $g : \omega^2 \to \{0, 1\}$ be as in the Definition 2. By the Church's Thesis, g is an intuitively computable total function. Devise an intuitive infinite procedure for A, satisfying the Definition 1. Let $x \in \omega$. Set $t = 0$. In the infinite loop do: intuitively compute $g(t, x)$, output the result in case it differs from the result obtained previously, increment t. This shows $A \in \mathcal{IL}$.

($\mathcal{IL} \subseteq \Delta^0_2$) Let $A \in \mathcal{IL}$. Then there is an intuitive algorithm, say G, satisfying the Definition 1. Without loss of generality, G never stops. Devise an intuitive algorithm G' that takes (t, x) as an input and returns the last answer generated by G on the input x up to t steps of the intuitive computation. By the Church's Thesis, the function intuitively computed by G' is recursive. Let g be that function. Clearly, g is total and satisfies the Definition 2. Hence, by the Limit Lemma, A is Δ^0_2. □

The main theorem of this paper (Theorem 6) states that the reverse implication does not hold. Before we give a proof, we analyse the proof of the Church's Thesis presented by Mostowski [6]. The proof of Mostowski goes in the direction that the Theorem 6 considers impossible. Of course, the proof of Mostowski uses some additional assumptions. We carefully discuss them and indicate their weak points.

3 Analysis of the Proof of Mostowski

In [6] M. Mostowski gives an argument for the Church's Thesis. The argument is based on three assumptions. **Ontological assumption**: there exist finitely, but potentially infinitely many objects. **Semantical assumption**: satisfaction and truth relations in finite models are recursive. **Epistemological assumption**: there exists a recursive enumeration of the FM-domain. It is namely assumed that cognitively accessible reality is finite, but potentially infinite, that our knowledge is expressible in our language and that it is decidable whether a given (without loss of generality - arithmetical) formula is satisfied in a finite, but sufficiently large (arithmetical) model and that enlarging the domain of the finite model we perform the computations (more generally: cognitive activity) in is recursive. Further, it is argued by the FM-representability theorem that the class of concepts that may be meaningfully described in a potentially infinite domain with recursive truth relation and recursive enumeration of finite approximations of the model is identical to the class of Δ^0_2 sets. Finally, an epistemological criterion separates computable relations from other FM-representable ones. The key notion employed in Mostowski's justification of the Church's Thesis is the notion of a **testing formula**.

Definition 4 (**Testing Formula**). *Let $R \subseteq \omega^n$ and $\varphi(x_1, \ldots, x_n)$ be a formula. A formula $\psi(x_1, \ldots, x_n)$ is a testing formula for $\varphi(x_1, \ldots, x_n)$ and R if:*

- *for each $a_1, \ldots, a_n \in \omega$ there is $n_0 \in \omega$ such that for each finite model M, $M \models \psi(a_1, \ldots, a_n)$ if and only if $|M| \geq n_0$,*

- *for each $a_1, \ldots, a_n \in \omega$ and each finite model M, if $M \models \psi(a_1, \ldots, a_n)$, then $R(a_1, \ldots, a_n)$ if and only if $M \models \varphi(a_1, \ldots, a_n)$.*

The conditions defining the notion of a testing formula for φ and R may be read as an explication of the concept of *knowing the answer (and achieving the answer effectively)* to the query of the form: *is a tuple $a_1, \ldots, a_n$ in the relation R?* Testing formulae then serve the abovementioned epistemological criterion of separating decidable relations from other FM-representable notions. This is justified by the following theorem.

Theorem 2 (Mostowski [7])**.** *Let $R \subseteq \omega^n$. R is decidable if and only if there are formulae $\varphi(x_1, \ldots, x_n)$, $\psi(x_1, \ldots, x_n)$ such that $\psi(x_1, \ldots, x_n)$ is a testing formula for $\varphi(x_1, \ldots, x_n)$ and R.*

Proof. Fix $R \subseteq \omega^n$.

($\Rightarrow$) Let $T(e, x_1, \ldots, x_n, c)$ be the Kleene predicate meaning that c is the code of the computation of the algorithm with code e on input $x_1, \ldots, x_n$ (note that every quantifier occurring in T is bounded by c). Let $U(c, y)$ mean that a computation with code c accepts if $y = 1$ or rejects if $y = 0$. Suppose that R is decidable and let e be the code of an algorithm deciding R. We define:

$$\psi(x_1, \ldots, x_n) = \exists c\, T(e, x_1, \ldots, x_n, c),$$

$$\varphi(x_1, \ldots, x_n) = \exists c\, (T(e, x_1, \ldots, x_n, c) \wedge U(c, 1)).$$

Fix $\overline{a} = a_1, \ldots, a_n \in \omega$. We show that ψ is a testing formula for φ and R. We have $\mathbb{N} \models \exists c\, T(e, \overline{a}, c)$ thus for some $n_0 \in \omega$ it holds that $\mathbb{N} \models T(e, \overline{a}, n_0)$. Since the computation of e on $\overline{a}$ is unique, so is n_0. Therefore for $m \in \omega$, $\mathbb{N}_m \models \psi(\overline{a})$ if and only if $m \geq n_0$.

Now fix $m \in \omega$ such that $\mathbb{N}_m \models \psi(\overline{a})$. Let $n_0 \in \omega$ be such that $\mathbb{N} \models T(e, \overline{a}, n_0)$. Then for every $m \geq n_0$ it holds that $\mathbb{N}_m \models T(e, \overline{a}, n_0)$. If $R(\overline{a})$, then $\mathbb{N} \models U(n_0, 1)$ and $\mathbb{N}_m \models \varphi(\overline{a})$. On the other hand if $\neg R(\overline{a})$, then $\mathbb{N} \models U(n_0, 0)$ and $\mathbb{N}_m \models \neg\varphi(\overline{a})$.

Therefore $\psi(x_1, \ldots, x_n)$ is a testing formula for φ and R.

($\Leftarrow$) Let $\psi(x_1, \ldots, x_n)$ be a testing formula for $\varphi(x_1, \ldots, x_n)$ and R. The algorithm deciding R is the following.

Algorithm 1 Algorithm deciding R

Input: $a_1, \ldots, a_n \in \omega$
Output: truth value of $R(a_1, \ldots, a_n)$

$i \leftarrow 0$
while $\mathbb{N}_i \not\models \psi(a_1, \ldots, a_n)$ **do**
    $i \leftarrow i + 1$
end while
return truth value of $\mathbb{N}_i \models \varphi(a_1, \ldots, a_n)$

The algorithm implicitly uses subroutines to compute $i \mapsto \ulcorner \mathbb{N}_i \urcorner$ and $\mathbb{N}_i \models \alpha$ which are both recursive. It also always halts since $\psi(x_1, \ldots, x_n)$ is a testing formula for $\varphi(x_1, \ldots, x_n)$ and R. This ends the proof. □

It is clear now that the Theorem 2 enables to identify recursive relations as the class for which we are able to *know* the model in which the truth of the relation's representing formula fixes. As we see, the proof of the Theorem 2 depends on two following statements:

1. There is a recursive enumeration of finite models,

2. Every finite model $\mathbb{N}_m$ has a recursive satisfaction relation.

While the second assumption is not controversial we take a closer look at 1. This takes us directly to key considerations needed in the proof of the Theorem 6. It is worth noting that the main assumptions of Mostowski's argument (namely the abovementioned ontological one and semantical one) taken together with the FM-representability theorem are actually equivalent to a version of the Learnability Thesis. It is so, since by those assumptions we model relations that can be meaningfully described in potentially infinite by an appropriate growing sequence of finite models with **computable** satisfaction relation. To put it in an even stronger way, one might say that any formal model compatible with ontological and semantical assumptions of Mostowski (which by the way seem to be plausible philosophical statements in general) shall be a class of finite models such that *meaningful* concepts are computed in the limit. In particular, such semantics gives us a class of formulae *decidable in the limit*, i.e. such that their interpretations stabilise after finitely many steps within an (potentially) infinite trial-and-error **computable** procedure. Such formulae express exactly intuitively learnable concepts. By the FM-representability theorem the set of such concepts is identical to the set of Δ^0_2 relations.

4 Learnability Thesis Does Not Entail Church's Thesis

So far, we have worked in relational arithmetical vocabulary $\sigma = (R_+, R_\times)$. Now we extend it to $\sigma' = \sigma \cup \{A\}$, where A is an additional 1-place predicate.

Theorem 3. *Let* $(\mathbb{N}, A)$ *be any* σ'*-model,* $R \subseteq \omega^n$*.* R *is decidable in* A *if and only if there are* σ'*-formulae* $\varphi(x_1, \ldots, x_n)$*,* $\psi(x_1, \ldots, x_n)$ *such that* $\psi(x_1, \ldots, x_n)$ *is a testing formula in* $\mathrm{FM}((\mathbb{N}, A))$ *for* $\varphi(x_1, \ldots, x_n)$ *and* R*.*

Proof. The proof is an easy generalisation of the proof of the Theorem 2.

($\Rightarrow$) It suffices to consider the Kleene predicate $T^A(e, x_1, \ldots, x_n, c)$ for oracle machines, meaning that c is the code of the computation of the oracle algorithm with Gödel number e on input $x_1, \ldots, x_n$ using A as an oracle.

($\Leftarrow$) The algorithm deciding R is essentially the same as the one from the proof of the Theorem 2, but since the map $i \mapsto \ulcorner(\mathbb{N}_i, A^{(i)})\urcorner$ is recursive in A, R is recursive in A (rather than just recursive as in the original proof). □

Taking $\mathrm{FM}(\mathbb{N})$ as our formal model is aimed at distinguishing exactly those properties that are essential for performing intuitive computations. It seems that considering the FM-domain of the finite cuts of an arithmetical model in which all predicate symbols have recursive interpretations, just as in case of $\mathrm{FM}(\mathbb{N})$, is actually equivalent to assuming that intuitively computable relations are exactly recursive ones, namely the Church's Thesis itself. Observe that if we admit the existence of some non-recursive but intuitively computable relations, we could intuitively compute the function $i \mapsto \ulcorner(\mathbb{N}_i, A^{(i)})\urcorner$ and by the theorem 2 exactly those relations which are recursive in A have testing formulae.

The arithmetical hierarchy can be naturally relativised to capture notions concerning computations relative to oracles. By extending the arithmetical vocabulary by an additional predicate and interpreting it as an oracle we obtain a relativised arithmetical hierarchy of definable notions relative to the oracle. A relation R is Δ^A_2 if it is definable both by Σ^A_2 and Π^A_2 formulae i.e.:

$$R(\overline{a}) \equiv \exists x \forall y\, P(x, y, \overline{a}), \tag{3}$$

$$R(\overline{a}) \equiv \forall x \exists y\, S(x, y, \overline{a}), \tag{4}$$

for some recursive in A predicates P and S. The following theorem is obvious by the relativisation of the Limit Lemma:

Theorem 4. *Let $R \subseteq \omega^n$. Then R is* FM*-representable in* $\mathrm{FM}((\mathbb{N}, A))$ *if and only if R is Δ^A_2.*

Definition 5 (Low Sets). *Let $A \subseteq \omega$. A is low if* $\deg(A)' = 0'$.

Of course every recursive set is low, but the converse does not hold. The existence of non-recursive low sets is a folklore (see for example [2]).

Theorem 5. *Let A be a low set. Then $\Delta^A_2 = \Delta^0_2$.*

Proof. Fix a low set A. The non-obvious inclusion is $\Delta^A_2 \subseteq \Delta^0_2$.

Fix a Δ^A_2 relation R. Then for some recursive in A predicates P and S we have:

$$R(\overline{a}) \equiv \exists x \underbrace{\forall y\, P(x, y, \overline{a})}_{\leq \deg(A)'}, \tag{5}$$

$$R(\overline{a}) \equiv \forall x \underbrace{\exists y\, S(x, y, \overline{a})}_{\leq \deg(A)'}. \tag{6}$$

Since A is low, $\deg(A)' = 0'$. Therefore by the generalised Post's theorem R is recursive in $0'$ and thus, by the Limit Lemma, R is Δ^0_2. □

Now, by an easy application of Theorems 4 and 5, we obtain:

Corollary 1. *Let A be a low set and $R \subseteq \omega^n$. Then R is FM-representable in* $\mathrm{FM}((\mathbb{N}, A))$ *if and only if R is Δ^0_2.*

By the Corollary 1, adding any low set A to the FM-domain does not affect the class of FM-representable relations and therefore the Learnability Thesis itself.

We are ready to prove our main theorem:

Theorem 6. *The Learnability Thesis does not entail the Church's Thesis.*

Proof. Let A be a low, non-recursive set. Let the interpretation of $\mathcal{IC}$ be $\{R : R \leq_T A\}$. Therefore under such an interpretation the Church's Thesis fails. On the other hand consider an FM-domain $\mathrm{FM}((\mathbb{N}, A))$. We may consider such an FM-domain since $A \in \mathcal{IC}$. By the Corollary 1 relations FM-representable in $\mathrm{FM}((\mathbb{N}, A))$ are exactly those which are Δ^0_2. Therefore the Learnability Thesis holds in such a model. We have shown that there is an interpretation of $\mathcal{IC}$ such that $\mathcal{IC} \neq \Delta^0_1$ and $\mathcal{IL} = \Delta^0_2$. Therefore the Learnability Thesis does not entail Church's Thesis. □

5 Concluding Remarks

In this paper we have described the Learnability Thesis and argued that an attempt of justifying the Church's Thesis based only on the Learnability Thesis must fail, by the Theorem 6. The clue of the argument is that there exists an interpretation of intuitive computability consistent with the Learnability Thesis such that certain intuitively computable sets are by no means recursive.

One of the paths of criticism towards our main result could proceed by questioning the *naturality* of our interpretation of $\mathcal{IC}$, namely that it is only theoretically admissible.[6] This is why we have performed the proof of the Theorem 6 in the framework that Mostowski used in his argument. This enabled us to justify the *naturality* of the interpretation of $\mathcal{IC}$ as $\{R : R \leq_T A\}$, for some low set A.[7] Mostowski used a very natural notion of a testing formula to show that recursive relations are exactly those FM-representable relations (equivalently - intuitively learnable) which have testing formulae. We have pointed out a flaw in his argument to show that if we admit some non-recursive but intuitively computable relations we are able to consider FM-domains expanded with their interpretations. This has led to singling out the relations recursive in A as those which have testing formulae in $\mathrm{FM}((\mathbb{N}, A))$. On the other hand, by the Corollary 1, relations FM-representable in such FM-domain are still Δ^0_2.

[6]The discussion on the naturality of the interpretation of $\mathcal{IC}$ started with our first attempt to prove that the Learnability Thesis does not entail Church's Thesis in which we considered $\mathcal{IC} = \{R : R \leq_T A,\ \text{for any low set } A\}$. Such an interpretation of intuitive computability, however, would have very unnatural properties since for instance there are low sets A, B such that their recursive sum $A \oplus B$ is Turing-equivalent to $0'$. Therefore a very natural operation such as taking a recursive sum of some two intuitively computable sets would lead to intuitively non-computable set (assuming the Learnability Thesis).

[7]Under such an interpretation, $\mathcal{IC}$ is closed under Turing-reducibility and therefore also under recursive sums, hence it is more *natural*.

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