Some Remarks on Least Moduli

Dariusz Kalociński*
Institute of Philosophy
University of Warsaw
Krakowskie Przedmieście 3
00-927 Warsaw, Poland
d.kalocinski@uw.edu.pl

Abstract

Modulus of a computable approximation is a function which returns the number of a stage at which the approximation has already converged for its argument. The least modulus points at the earliest such stage for each of its arguments. We recall and show some properties of least moduli, including their close connection to c.e. degrees, and minimal witnessing functions for *FM*-representable sets. We observe, for instance, that the non-density theorem for the d.c.e. degrees gives an example of an incomplete degree that has no least moduli below $\underline{0}'$. Using the properties of least moduli themselves, we construct a degree containing no least moduli for itself and having least moduli of incomparable degrees. In particular, the technique used demonstrates an approach of constructing a non-c.e. degree, which is somewhat different from that proposed by Cooper.

keywords modulus, witnessing function, c.e. degrees, d.c.e. degrees, non-c.e. degrees, incomparable degrees

1 Introduction

A family of sets of integers $(A_s)_{s\in\omega}$ is said to be a computable sequence of computable sets if there is a total computable function $f : \omega^2 \to \{0,1\}$ such that $A_s(x) = f(x,s)$, for all $x, s \in \omega$. We say that $(A_s)_{s\in\omega}$ is a computable approximation if $(A_s)_{s\in\omega}$ is a computable sequence of computable sets and for each $x \in \omega$ the set $\{s : A_s(x) \neq A_{s+1}(x)\}$ is finite. When $(A_s)_{s\in\omega}$ is a computable approximation then, for each $x \in \omega$, it is meaningful to speak about the limit $\lim_{s\to\infty} A_s(x)$. For such a family, there is a unique set $A \subseteq \omega$ defined by $A = \lambda x[\lim_{s\to\infty} A_s(x)]$. A thus defined is said to be the limit of $(A_s)_{s\in\omega}$, and is often written as $\lim_s A_s = A$. We also use a short form (A_s) instead $(A_s)_{s\in\omega}$.

Shoenfield [1] proved that sets $\leq \underline{0}'$ are precisely those which have computable approximations:

Theorem 1 (Limit Lemma, Shoenfield [1])**.** $A \leq \underline{0}'$ *iff there is a computable approximation* (A_s) *such that* $\lim_s A_s = A$.

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This class is nicely captured by other formalisms as well: it coincides with Δ^0_2-definable [2, 3] and *FM*-representable sets [4]. When thinking about computable approximations, it is natural to ask about the properties of their moduli (of convergence) [1].

Definition 1. *$f : \omega \to \omega$ is a modulus of a computable approximation (A_s) if for every $x \in \omega$:*

$$\forall t \geq f(x)\, A_t(x) = A(x). \tag{1}$$

A related notion is that of minimal modulus.

Definition 2. *f is a minimal modulus of a computable approximation (A_s) if for every $x \in \omega$:*

$$f(x) = \mu s\, [\forall t \geq s\, A_t(x) = A(x)]. \tag{2}$$

Strictly speaking, a modulus (or a minimal modulus) is always a function associated with some computable approximation. However, we shall often speak about moduli or minimal moduli of a set or even of a degree. This should be understood as follows. A function f is a (minimal) modulus of A if there is a computable approximation (A_s) such that $\lim_s A_s = A$ and f is the (minimal) modulus of (A_s). Moreover, f is a (minimal) modulus of a degree $\underline{\mathrm{a}}$ if there is $A \in \underline{\mathrm{a}}$ such that f is a (minimal) modulus of A. In the context of *FM*-representability, the corresponding concepts are that of *witnessing function* and *minimal witnessing function* [4]. Before we introduce them, let us briefly recapitulate the basics of *FM*-representability.

Let $\mathbb{N} = (\omega, +, \cdot)$ be the standard model of PA, where $+$ and $\cdot$ are interpreted as ternary relations. $\mathbb{N}_k$ denotes $(\omega \restriction k, + \restriction k, \cdot \restriction k)$, for $k > 0$. We take arithmetical formulae to be first-order formulae over the relational vocabulary $(+, \cdot)$. We say an arithmetical formula $\varphi(x)$ *FM*-represents $A \subseteq \omega$ if for every n there is t such that for all $s \geq t$: $A(n) \Leftrightarrow \mathbb{N}_s \models \varphi(n)$. $A \subseteq \omega$ is *FM*-representable if there is an arithmetical formula $\varphi(x)$ *FM*-representing A. Suppose $\varphi(x)$ *FM*-represents a set. We say that $f : \omega \to \omega$ is a witnessing function for φ if $\forall n\, \forall s \geq f(n)\, [\mathbb{N}_{f(n)} \models \varphi(n) \iff \mathbb{N}_s \models \varphi(n)]$. Moreover, $f : \omega \to \omega$ is a minimal witnessing function for φ if $\forall n\, f(n) = \mu s[\forall t \geq s\, (\mathbb{N}_s \models \varphi(n) \iff \mathbb{N}_t \models \varphi(n))]$. By the *FM*-representability theorem [4], *FM*-representable sets are precisely those which are $\leq \underline{0}'$.

A (minimal) witnessing function is always associated with a particular *FM*-representing formula. However, we often speak about witnessing or minimal witnessing functions of sets and degrees. We believe the reader is able to make these concepts clear (as we have done it for moduli).

Given $\varphi(x)$ *FM*-representing A, we can define a computable approximation (A_s) such that $\lim_s A_s = A$ by setting $A_s(n) = 0$, if $s \leq n$, and $A_s(n) =$ the truth value of $\mathbb{N}_s \models \varphi(n)$, if $s > n$. In a sense, one can view *FM*-representing formulae as computable approximations. However, it is not the case that every computable approximation $\lim_s A_s = A$ has a corresponding arithmetical formula $\varphi(x)$ *FM*-representing A and satisfying $A_s(n) \Leftrightarrow \mathbb{N}_s \models \varphi(n)$ (whenever this is meaningful). This can be easily observed by applying a diagonal argument: take any computable approximation (A_s), a computable enumeration $\varphi_0(x), \varphi_1(x), \ldots$ of arithmetical formulae with one free variable and define a computable approximation (A'_s) by setting $A'_s(n) = A_s(n)$, if $s \neq n+1$ and $A'_s(n) =$ the truth value of $\mathbb{N}_s \models \neg\varphi_s(n)$ if $s = n+1$. Taking $s = n+1$ is for the most part arbitrary: this is the smallest s for which $\mathbb{N}_s \models \neg\varphi(n)$ is meaningful.

The above property should not worry us too much because we primarily want to look at the Turing degrees of minimal moduli and minimal witnessing functions. As we shall see in Section 2, the degrees of minimal moduli and minimal witnessing functions are the same. This gives us an opportunity to examine the degrees of both kinds of functions by analysing the properties of minimal moduli themselves.

2 Basic properties

In this section, we give some basic properties of least moduli and their Turing degrees.

Proposition 1 (Modulus Lemma, Shoenfield [1]). *Let $A \leq B$, where B is c.e. Then there is a computable approximation (A_s) and a function m such that $\lim_s A_s = A$, m is a modulus of (A_s) and $m \leq B$.*

We introduce a notion of a self-modular degree. This notion does not appear in the literature, but will simplify our presentation, saving us from saying "(not) having a modulus for itself" (with reference to sets and degrees) repeatedly.

Definition 3. *A is self modular if there is a modulus m of A such that $m \leq A$. A degree is self modular if it contains a self-modular set.*

The following proposition is a corollary of Proposition 1 and the Limit Lemma (Theorem 1). It shows that c.e. degrees and self-modular degrees coincide.

Proposition 2 (Soare [5]). *$\underline{a}$ is c.e. iff $\underline{a}$ is self modular.*

As for least moduli, we can easily prove the following:

Proposition 3. *For any computable approximation (A_s), the least modulus of (A_s) is of c.e. degree.*

Proof. Let m be the least modulus of a computable approximation (A_s). Define a c.e. set $M = \{(x, s) : \exists\, t \geq s\, A_{t+1}(x) \neq A_t(x)\}$ and show that $m \equiv M$. □

As far as witnessing functions are concerned, Mostowski [4] shows that a set is *FM*-representable iff it has a witnessing function $\leq \underline{0}'$. By an easy modification of the previous argument, we can obtain

Proposition 4. *For any FM-representing formula $\varphi(x)$, the minimal witnessing function of $\varphi(x)$ is of c.e. degree.*

Given a set $A \leq \underline{0}'$ and its modulus m, we always have $A \leq m$. Hence, the degree of any least modulus m is to be found among c.e. degrees $\geq A$. The question is whether one can find a least modulus of A in any c.e. degree $B \geq A$. It is not hard to prove that this is in fact the case. One can do this by constructing an appropriate computable approximation to A. However, we approach this question in a slightly different way. We prove that each *FM*-representable set A has minimal witnessing functions in all c.e. degrees $\geq A$ (Proposition 5). The result for computable approximations and least moduli will follow (Proposition 6).

Before we proceed to the proof, let us introduce some notation. $<$ is defined as usual. For an arithmetical formula ϕ and a definable constant k, write $\phi_{<k}$ for ϕ with all quantifiers strictly bounded by k.[1] Observe that there is a simple formula that defines a constant *MAX* which, in each initial segment of $\mathbb{N}$, denotes its unique maximal element. The following simple lemma will be useful.

[1] This can be made precise as follows. If ϕ is $x = y$, $x + y = z$ or $x \times y = z$ then $\phi_{<k}$ is ϕ. If ϕ is $\neg\psi$ or $\psi \Rightarrow \xi$ then $\phi_{<k}$ is $\neg\psi_{<k}$ or $\psi_{<k} \Rightarrow \xi_{<k}$, respectively. If ϕ is $\exists x\, \psi$ then $\phi_{<k}$ is $\exists x < k\, \psi_{<k}$.

Lemma 1. *Let $\phi(\overline{x})$ be an arithmetical formula. For all $n \in \omega$, $\overline{a} < n$: $\mathbb{N}_n \models \phi(\overline{a})$ iff $\mathbb{N}_{n+1} \models \phi_{<MAX}(\overline{a})$.*

Proposition 5. *Suppose $A \leq B$ and B is c.e. Then, there is a minimal witnessing function of A of the same degree as B.*

Proof. Choose machines a and b such that A is the set computed by a in B, and $B = \{x \in \omega : \exists c\, T(b, x, c)\}$, where T is the Kleene predicate. Without loss of generality, assume $0 < a < b$. Let $\beta(x) := \exists c\, T(b, x, c)$. Let T^X be the Kleene predicate for machines with oracle. If ϕ is an arithmetical formula with one free variable, let T^ϕ denote the Kleene predicate T^X with all occurrences of $X(y)$ replaced by $\phi(y)$. Given a computation c, let $U(c)$ be its output. Now, define an arithmetical formula $\psi(x)$ as

$$[\exists c(c = MAX \wedge T(b, x, c)) \implies \neg\exists c < MAX(T^\beta_{<MAX}(a, x, c) \wedge U(c) = 1)] \wedge \tag{3a}$$

$$\wedge [\neg\exists c(c = MAX \wedge T(b, x, c)) \implies \exists c(T^\beta(a, x, c) \wedge U(c) = 1)] \tag{3b}$$

We show that $\psi(x)$ *FM*-represents A. Let $n \in \omega$. Observe that (3a) and the predecessor of (3b) are true in sufficiently large models. Hence, the successor of (3b) determines the truth value of $\psi(n)$. Sufficiently large models contain the computation of a on input n with all necessary questions to the oracle B answered correctly.[2] Therefore, in sufficiently large models, $\psi(n)$ is true exactly when $n \in A$.

Now, let m be the minimal witnessing function of $\psi(x)$. We show that $B \leq m$. To compute $B(n)$ return the truth value of $\mathbb{N}_{m(n)} \models \beta(n)$. This is obvious, if $n \notin B$ because $\mathbb{N}_{m(n)} \not\models \beta(n)$. Assume $n \in B$ and let c_0 be such that $T(b, n, c_0)$. Then $\mathbb{N}_{c_0+1} \models \exists c(c = MAX \wedge T(b, n, c))$. By inspecting $\psi(x)$ we see that $\mathbb{N}_{c_0+1} \models \psi(n)$ exactly when $\mathbb{N}_{c_0+1} \models \neg\exists c < MAX(T^\beta_{<MAX}(a, x, c) \wedge U(c) = 1)]$ (this is meaningful because $a < c_0$). By Lemma 1, $\mathbb{N}_{c_0+1} \models \neg\exists c < MAX(T^\beta_{<MAX}(a, x, c) \wedge U(c) = 1)]$ iff $\mathbb{N}_{c_0} \models \neg\exists c(T^\beta(a, x, c) \wedge U(c) = 1)]$. But $\mathbb{N}_{c_0} \models \psi(n)$ exactly when $\mathbb{N}_{c_0} \models \exists c(T^\beta(a, x, c) \wedge U(c) = 1)]$. Hence, $\mathbb{N}_{c_0} \models \psi(n) \Leftrightarrow \mathbb{N}_{c_0+1} \not\models \psi(n)$. Consequently, $m(n) \geq c_0 + 1$. Therefore, $\mathbb{N}_{m(n)} \models \beta(n)$.

Now, we show how to find $m(n)$ recursively in B. We simply look for $s_0 :=$ the least s such that $\mathbb{N}_s$ contains a, b, n, the computation of b on input n (if it exists) and the B-computation of a on input n. Obviously, $m(n) \leq s_0 + 1$. Now, finding the value of $m(n)$ is easy. □

Proposition 6. *Suppose $A \leq B$ and B is c.e. Then, there is a least modulus of A of the same degree as B.*

Proof. Let $a, b, \psi(x)$ be as in the proof of Proposition 5. Let m be the minimal witnessing function of $\psi(x)$. Define a computable approximation (A_s) as follows:

$$A_s(x) = \begin{cases} 0 & s < max(a, b, x) \quad (1) \\ \text{the truth value of } \mathbb{N}_{max(a,b,x)+1} \models \neg\psi(x) & s = max(a, b, x) \quad (2) \\ \mathbb{N}_s \models \psi(x) & s > max(a, b, x) \quad (3) \end{cases}$$

Here, max is a number-theoretic function selecting the maximal argument. Obviously, $A_0(x) = 0$ always (this is what we usually want from a computable approximation), and $\lim_s A_s = A$. We show that m is the least modulus of (A_s). $m(x)$ points to the least

[2] It is important to notice that, for any $k \in \omega$, when $\beta(k)$ becomes true in some model then it stays true in larger models. This property guarantees that when a question to the oracle is established for the first time, it remains established forever.

stage s such that the truth value of $\mathbb{N}_s \models \psi(x)$ is correct and remains correct later on. Hence, $\mathbb{N}_{m(x)} \models \psi(x)$ is a meaningful statement and, consequently, $m(x) > max(a, b, x)$. Moreover, $A_t(x) =$ the truth value of $\mathbb{N}_{m(x)} \models \psi(x)$, for all $t \geq m(x)$. Suppose $m(x) = max(a, b, x) + 1$. By (2) we have $A_{m(x)-1}(x) \neq A_{m(x)}(x)$. So $m(x)$ points to the least stage at which (A_s) settles down for x. Now, assume $m(x) > max(a, b, x) + 1$. Then, we have $\mathbb{N}_{m(x)-1} \models \psi(x) \not\Leftrightarrow \mathbb{N}_{m(x)} \models \psi(x)$ and thus $A_{m(x)-1}(x) \neq A_{m(x)}(x)$. Again, $m(x)$ is as wanted. □

Propositions 5 and 6 give us a broad picture of the Turing degrees of least moduli: for any degree $\underline{\mathtt{a}} \leq \underline{\mathtt{0}}'$, the degrees of its least moduli are precisely $\{\underline{\mathtt{b}} : \underline{\mathtt{b}} \geq A \wedge \underline{\mathtt{b}} \text{ is c.e.}\}$.

Finally, let us prove another fact which will be useful in our main construction from Section 4.

Proposition 7. *A degree is self modular iff all its members have least moduli precisely of that degree.*

Proof. The right-to-left part is obvious. Assume $\underline{\mathtt{a}}$ is self modular. Then, by Proposition 2, $\underline{\mathtt{a}}$ is c.e. Choose a c.e. set $B \in \underline{\mathtt{a}}$. Let $A \in \underline{\mathtt{a}}$. We have $A \leq B$. By Proposition 6, A has a least modulus of the degree $\underline{\mathtt{a}}$. □

3 Other properties

Natural questions arise in the context we have already outlined. For example, one may be tempted to find degrees the least moduli of which behave in essentially different ways. In this section, we give a few such examples.

Recall that $A \subseteq \omega$ is 2-c.e. (or d.c.e.) if there are c.e. sets A_1, A_2 such that $A = A_1 - A_2$. This notion can be generalized to any $n > 2$. Roughly speaking, a set is n-c.e. if it is a boolean combination of n c.e. sets. Such sets were introduced by Putnam [2] and Gold [3]. A Turing degree is n-c.e. if it contains some n-c.e. set. Cooper [6] was the first to separate $(n+1)$-c.e. and n-c.e. degrees (see also [7]):

Theorem 2 (Cooper [6])**.** *For all $n > 0$, there is a properly $(n+1)$-c.e. degree [i.e., an $(n+1)$-c.e. degree which is not n-c.e.].*

Observe that by Proposition 1, a properly $(n+1)$-c.e. degree does not contain any of its least moduli. All least moduli of such a set are to be found strictly above it. In contrast, by Proposition 6, each c.e. degree contains least moduli for all its members. Moreover, by the Sacks Density Theorem [8], given any c.e. degree $\underline{\mathtt{a}} < \underline{\mathtt{0}}'$, there is a degree $\underline{\mathtt{b}}$ such that $\underline{\mathtt{a}} < \underline{\mathtt{b}} < \underline{\mathtt{0}}'$. Hence, by Proposition 6, there are least moduli for $\underline{\mathtt{a}}$ strictly between $\underline{\mathtt{a}}$ and $\underline{\mathtt{0}}'$. However, this is not the case for $(n+1)$-c.e. degrees. To see this, we turn to the following powerful result:

Theorem 3 (Cooper, Harrington, Lachlan, Lempp and Soare [9])**.** *There is a maximal incomplete 2-c.e. degree (i.e., a 2-c.e. degree $\underline{\mathtt{d}} < \underline{\mathtt{0}}'$ such that there is no 2-c.e. degree $\underline{\mathtt{e}}$ with $\underline{\mathtt{d}} < \underline{\mathtt{e}} < \underline{\mathtt{0}}'$).*

Since every c.e. degree is also 2-c.e. (because it contains a 2-c.e. set), there can be no c.e. degrees strictly between $\underline{\mathtt{d}}$ and $\underline{\mathtt{0}}'$. Therefore, we obtain the following:

Corollary 1. *There is a degree $\underline{\mathtt{d}} < \underline{\mathtt{0}}'$ such that all its least moduli are of degree $\underline{\mathtt{0}}'$.*

What other configurations of c.e. degrees can we find between 2-c.e. degrees and $\underline{0}'$? This sort of question has received some attention in the context of maximality and splitting properties. For example, Arslanov, Cooper and Li [10, 11] showed that a maximal 2-c.e. degree cannot be low. Moreover:

Theorem 4 (Arslanov, Cooper and Li [10, 11])**.** *For any low 2-c.e. degree $\underline{d}$ there are incomparable c.e. degrees $\underline{a}_0, \underline{a}_1 > \underline{d}$ such that $\underline{a}_0 \cup \underline{a}_1 = \underline{0}'$.*

How can one obtain a properly 2-c.e. degree that is low? One way of doing so, perhaps not the most elegant, is to use Cooper's technique [6] for constructing a properly 2-c.e. degree and attach to it a permitting argument in a non-computable low c.e. degree (low c.e. degrees $> \underline{0}$ exist—for example, the original Friedberg-Muchnik construction [12, 13] yields such degrees, as observed by Soare [14]). In such a manner, we can obtain a properly 2-c.e. degree, say $\underline{d}$, that is low and, by Theorem 4, $\underline{0}'$ splits above $\underline{d}$ into two incomparable c.e. degrees. Hence, by Proposition 6, $\underline{d}$ is non self modular and has two incomparable least moduli.

In what follows, we construct a set of this kind. However, our construction achieves this goal differently: it exploits the properties of least moduli themselves and is based on requirements which explicitly refer to least moduli of the set under construction. This gives us another way of constructing a properly 2-c.e. degree, somewhat different from Cooper's technique [6].

4 A non-self-modular degree with incomparable least moduli

Theorem 5. *There is a degree which is not self modular and has least moduli of incomparable degrees.*

We construct a set F and functions f, g by full approximation: we build computable approximations $(F_s), (G_s), (f_s), (g_s)$ such that $\lim_s F_s = F = \lim_s G_s$, $\lim_s f_s = f$ and $\lim_s g_s = g$. To make $deg(F)$ non self modular, we use a strategy described in Section 4.2. To make $deg(f)$ and $deg(g)$ incomparable, we use a slightly modified Friedberg-Muchnik technique, as described in Section 4.1. This gives us $f \not\leq g$ and $g \not\leq f$. We also make f and g into the least moduli of (F_s) and (G_s), respectively. Hence, we have $F \leq f$ and $F \leq g$.

We use fixed enumerations of all partial computable functionals $\Psi_0, \Psi_1, \Psi_2, \ldots$, and all partial computable functions $\phi_0, \phi_1, \phi_2, \ldots$. We write $\Psi_{n,s}$ and $\phi_{n,s}$ for Ψ_n and ϕ_n enumerated up to stage s, respectively. The use function of a partial computable functional (approximated up to stage s) $\Phi^X_{e,s}(x)$ is denoted by the corresponding small greek letter $\varphi^X_{e,s}(x)$ and is equal to 1 plus the maximal number used in s steps of the computation of the algorithm e on input x relative to the oracle X, if the computation stops in $\leq s$ steps, and $= 0$ otherwise. $\varphi^X_e(x)$ is $\varphi^X_{e,s}(x)$ if $\Phi^X_{e,s}(x)\downarrow$, for some s, and undefined otherwise.

4.1 Making f and g incomparable

For the most part, this is Friedberg and Muchnik. There are three differences. First is that we construct functions which can assume arbitrarily large values. Second is that we try to make f and g into the least moduli of (F_s) and (G_s). Third is that we want to ensure that $\lim_s F_s = \lim_s G_s$. To handle the last requirement, we shall use pending functions $p_F(x,s)$, $p_G(x,s)$. When $p_F(x,s) = 1$ (or $p_G(x,s) = 1$) we have a pending action: withdraw x from F using (F_s) (or (G_s)) at stage $s+1$.

For each $e \in \omega$, we have the following requirements:

$$\begin{array}{ll} \mathcal{F}_e : & f \neq \Phi_e^g, \\ \mathcal{G}_e : & g \neq \Phi_e^f. \end{array}$$

We describe a strategy for meeting $\mathcal{F}_e$ in isolation (the strategy for $\mathcal{G}_e$ is the same, except obvious differences). $\mathcal{F}_e$ is assigned a witness x. We try to guarantee $f(x) \neq \Phi_e^g(x)$ by making $f_0(x) = 0, f_1(x) = 0, \ldots$ unless we encounter a stage t with $\Phi_{e,t}^{g_t}(x) \downarrow = 0$. If there is no such stage then either $\Phi_e^g(x) > 0$ or $\Phi_e^g(x) \uparrow$, leaving $\mathcal{F}_e$ satisfied. Let t_0 be the least stage with $\Phi_{e,t_0}^{g_{t_0}}(x) \downarrow = 0$. We freeze $g_{t_0} \upharpoonright \varphi_{e,t_0}^{g_{t_0}}(x)$ to preserve $\Phi_{e,t_0}^{g_{t_0}}(x) \downarrow$ and to have $\Phi_e^g(x) = \Phi_{e,t_0}^{g_{t_0}}(x)$. We set $f_{t_0+1}(x) = t_0 + 1, F_{t_0+1}(x) = 1$ and $p_F(x, t_0+1) = 1$. At stage $t_0 + 2$ we have a pending action: we withdraw x from F using (F_s). This forces us to set $f_{t_0+2}(x) = t_0 + 2$. Thanks to this mandatory withdrawal, we obtain $\lim_s F_s = \lim_s G_s$. Eventually, we get $f(x) = t_0 + 2 \neq 0 = \Phi_e^g(x)$.

4.2 Making a non-self-modular degree

In this section, we develop requirements for constructing a non-self-modular degree (observe that by Proposition 2, such a degree must be non-c.e). Given such a requirement, we also show how are we going to satisfy it (in isolation).

4.2.1 Requirements

By Proposition 7, a degree is non self modular iff it contains a set such that none of its least moduli is precisely of that degree. More formally, we want to construct F such that for all computable approximations (B_s) and all functions $m : \omega \to \omega$:

$$m \text{ is the least modulus of } (B_s) \ \wedge \ (B_s) \text{ converges to } F \implies m \not\leq F.^3 \tag{4}$$

Equivalently:

$$m \text{ is the least modulus of } (B_s) \ \wedge \ m \leq F \implies (B_s) \text{ does not converge to } F \tag{5}$$

To translate this into requirements for our construction, some more notation is in order. We say ϕ_e is a computable approximation if ϕ_e is total, assumes values in $\{0,1\}$ and for every $x \in \omega$, $\lim_{t\to\infty} \phi_e(x,t)$ exists. We write $\lim \phi_e$ to denote the partial function f defined as $f(x) := \lim_s \phi_e(x,s)$ if $\lambda s[\phi_e(x,s)]$ is total and $\lim_s \phi_e(x,s)$ exists, undefined otherwise. $\lim \phi_e$ may be viewed as a (partial) characteristic function. Now, in terms of functionals, Ψ_n^F is the least modulus of ϕ_e iff Ψ_n^F is total, ϕ_e is a computable approximation and $\Psi_n^F = \lambda x[\mu s(\forall t \geq s\, \phi_e(x,t) = \lim_u \phi_e(x,u))]$. Hence, one can reformulate (5) and obtain individual requirements which we apply in the construction: $deg(F)$ is non self modular iff $\mathcal{R}_{e,n}$ holds for all $e, n \in \omega$, where:

$$\mathcal{R}_{e,n} : \Psi_n^F \text{ is the least modulus of } \phi_e \Rightarrow \lim \phi_e \neq F \tag{6}$$

[3] Recall that if m is a least modulus for F then $F \leq m$.

4.2.2 Strategy

How are we going to satisfy a single $\mathcal{R}_{e,n}$? We reserve for $\mathcal{R}_{e,n}$ a fresh witness x (in particular, x is not enumerated into F at this point) and wait for a stage s such that $\Psi^{F_s}_{n,s}(x) \downarrow= m$ (if we always have $\Psi^{F_s}_{n,s}(x) \uparrow$ then our requirement remains satisfied because Ψ^F_n is not a least modulus). Suppose we have $\Psi^{F_s}_{n,s}(x) \downarrow$ at some stage s and let $m = \Psi^{F_s}_{n,s}(x)$. We preserve the computation with the restraint $\psi^{F_s}_{n,s}(x)$ and wait for a stage $t > s$ such that $\phi_{e,t}(x, m) \downarrow$ (if there is no such stage, $\mathcal{R}_{e,n}$ remains satisfied forever because ϕ_e is not a computable approximation). Suppose we have $\phi_{e,t}(x, m) \downarrow$ at some stage $t \geq s$ and let $\phi_{e,t}(x, m) = 1$. In this case, we do not take any action. Let us see why not taking any action is a good option. If ϕ_e is a computable approximation and settles down beginning at stage m, then $\lim_{t\to\infty} \phi_e(x, t) = 1$. However, x has not been enumerated in F at any previous stage, so we have $F_t(x) = 0$. If x stays out of F later on, we have $F(x) = 0$. Hence, $\lim_{t\to\infty} \phi_e(x, t) \neq F(x)$. If ϕ_e is a computable approximation and does not settle down at stage m then it settles down at a later stage and thus Ψ^F_n is not the least modulus for ϕ_e and $\mathcal{R}_{e,n}$ remains satisfied. Obviously, if ϕ_e is not a computable approximation, $\mathcal{R}_{e,n}$ remains satisfied as well.

Now, let us consider the case when we have a restraint $\psi^{F_s}_{n,s}(x)$ and a later stage $t > s$ with $\phi_{e,t}(x, m) = 0$. If we do not take any action now, we may be in trouble. If Ψ^F_n is the least modulus of ϕ_e then we have $\lim_{t\to\infty} \phi_e(x, t) = F(x)$, a situation we want to avoid. Therefore, we set $F_{t+1}(x) = 1$. The restraint $\psi^{F_s}_{n,s}(x)$ is still there. Now, for $u \geq t + 1$, the computation $\Psi^{F_u}_{n,u}(x)$ may be different from $\Psi^{F_s}_{n,s}(x)$. Hence, we wait for a stage $u > t$ such that $\Psi^{F_u}_{n,u}(x) \downarrow$. Again, our requirement will remain satisfied if there is no such stage since then Ψ^F_n is not the least modulus of ϕ_e. Let $u > t$ be such that $\Psi^{F_u}_{n,u}(x) \downarrow= m'$. We wait for stage $v \geq u$ with $\phi_{e,v}(x, m') \downarrow$. Our requirement is satisfied if there is no such stage since then ϕ_e is not a computable approximation. Let $v \geq u$ be such that $\phi_{e,v}(x, m') \downarrow$. We have two cases, either $\phi_e(x, m') = 0 \vee (\phi_e(x, m') = 1 \wedge m' < m)$ or $\phi_e(x, m') = 1 \wedge m' \geq m$.

First, assume $\phi_e(x, m') = 0 \vee (\phi_e(x, m') = 1 \wedge m' < m)$. Then, we take no action. Let us see why it works. First, suppose $\phi_e(x, m') = 0$ and assume Ψ^F_n is the least modulus of ϕ_e. We have $\lim_t \phi_e(x, t) = 0 \neq F(x)$, so $\lim \phi_e \neq F$. Second, assume $\phi_e(x, m') = 1 \wedge m' < m$. We have $\phi_e(x, m') = 1$, $\phi_e(x, m) = 0$ and $m' < m$, so Ψ^F_n is not the least modulus of ϕ_e. Now, consider the case $\phi_e(x, m') = 1 \wedge m' \geq m$. In fact, we have $m' > m$, because $\phi_e(x, m) = 0$. Therefore, we set $F_{v+1}(x) = 0$. Now, because the restraint $\psi^{F_s}_{n,s}(x)$ is still valid and $F_s(x) = F_{v+1}(x) = 0$, $\Psi^{F_{v+1}}_{n,v+1}(x) = \Psi^{F_s}_{n,s}(x) = m$. We have $\phi_e(x, m) = 0$, $\phi_e(x, m') = 1$ and $m < m'$. Consequently, Ψ^F_n is not the least modulus of ϕ_e.

We enclose our findings in a concise condition which specifies when we should act in order to satisfy $\mathcal{R}_{e,n}$ in isolation. We use an auxiliary function $h(e, n, x, s)$—the modulus tracker—which is set at the beginning (i.e., for $s = 0$) to 0 for all arguments e, n, x. It will be convenient to write $h_s(e, n, x)$ instead of $h(e, n, x, s)$. Recall that we decide to act for the first time when we encounter a stage $t+1$ with $m := \Psi^{F_t}_{n,t}(x) \downarrow$ and $\phi_{e,t}(x, m) = 0$. Observe that this is exactly when the condition $\phi_{e,t}(x, \Psi^{F_t}_{n,t}(x)) = F_t(x) \wedge \Psi^{F_t}_{n,t}(x) \geq h_t(e, n, x)$ is satisfied: since x is the fresh witness, we have $F_t(x) = 0$ and thus $\phi_{e,t}(x, m) = 0$; obviously, $m \geq h_t(e, n, x)$ because the modulus tracker is set to 0 from the start and repeats its previous value when nothing happens. When acting for the first time, say at stage $t + 1$, we enumerate x into F and keep the restraint for $\Psi^{F_t}_{n,t}(x)$. This is where we update the modulus tracker: $h_{t+1}(e, n, x) = m$. According to our considerations from this section, we do not have to take any further actions unless we encounter a stage $u > t$

with $m' := \Psi^{F_u}_{n,u}(x)\downarrow$, $\phi_{e,u}(x, m') = 1$ and $m' \geq m$. This is precisely when the condition $\phi_{e,u}(x, \Psi^{F_u}_{n,u}(x)) = F_u(x) \wedge \Psi^{F_u}_{n,u}(x) \geq h_u(e, n, x)$ is satisfied.

4.3 Main construction

Let us recall our requirements:

$$\begin{array}{ll} \mathcal{F}_e : & f \neq \Phi^g_e \\ \mathcal{G}_e : & g \neq \Phi^f_e \\ \mathcal{R}_{e,n} : & \Psi^F_n \text{ is the least modulus of } \phi_e \Rightarrow \lim \phi_e \neq F \end{array}$$

Write $\mathcal{R}_k$ interchangeably with $\mathcal{R}_{k_1,k_2}$, where k is the code of (k_1, k_2). Fix a computable priority ordering, for example: $\mathcal{F}_0 < \mathcal{G}_0 < \mathcal{R}_0 < \mathcal{F}_1 < \mathcal{G}_1 < \mathcal{R}_1 < \mathcal{F}_2 < \mathcal{G}_2 < \mathcal{R}_2 < \dots$. As usual, stronger priorities are to the left.

We say $\mathcal{F}_e$ requires attention at stage $s+1$ if there is an x reserved for $\mathcal{F}_e$ at stage $s+1$ and $f_s(x) = \Phi^{g_s}_{e,s}(x)$. Similarly, we say $\mathcal{G}_e$ requires attention at stage $s+1$ if there is an x reserved for $\mathcal{G}_e$ at stage $s+1$ and $g_s(x) = \Phi^{f_s}_{e,s}(x)$. We say that $\mathcal{R}_{e,n}$ requires attention at stage $s+1$ if there is an x reserved for it at stage $s+1$ and $\phi_{e,s}(x, \Psi^{F_s}_{n,s}(x)\downarrow) = F_s(x) \wedge \Psi^{F_s}_{n,s}(x)\downarrow \geq h_s(e, n, x)$.

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Stage 0. Set $f_0 = g_0 = F_0 = G_0 = \emptyset$ and $p_F(x, 0) = p_G(x, 0) = 0$, for all $x \in \omega$, which means that there are no pending actions associated with functions f, g and approximations (F_s), (G_s). We also set the modulus tracker $h_0(e, x, n) = 0$, for all arguments e, x, n. No number is reserved for any requirement. There are no restraints associated with any requirement either.

Stage $s+1$. See if there are any pending actions, i.e., $p_F(x, s)$ or $p_G(x, s)$ are > 0 for some x. This is a recursive question because pending functions can assume values > 0 only for numbers that have been already reserved for some requirements and there is only a finite number of such numbers and we know their least upper bound at each stage. If there are some pending actions, we perform them. In fact, at any stage we always have only one pending action, if any. So, consider a case where we have a pending action $p_F(e, x) = 1$. We set $F_{s+1}(x) = 0$, $f_{s+1}(x) = s+1$, $p_F(e, x) = 0$ and proceed to the next stage. If we have $p_G(e, x) = 1$ then we set $G_{s+1}(x) = 0$, $g_{s+1}(x) = s+1$, $p_G(e, x) = 0$ and proceed to the next stage.

If there are no pending actions [i.e., $p_F(x, s) = p_G(x, s) = 0$ for all $x \in \omega$], check whether there are any requirements needing attention. If there are none, then take the strongest requirement among those which have no reservations and reserve for it a fresh witness. Proceed to the next stage.

Suppose there are no pending actions and some requirements need our attention. Take the strongest such requirement—we say it receives our attention. Let x be the number reserved for it at the present stage. Cancel all restraints and reservations for requirements of lesser priority.

Assume that the requirement receiving our attention is $\mathcal{F}_e$ (we handle $\mathcal{G}_e$ in a similar way). Hence, we have $f_s(x) = \Phi^{g_s}_{e,s}(x)$. Preserve the computation $\Phi^{g_s}_{e,s}(x)$ with the restraint $\varphi^{g_s}_{e,s}(x)$. Put $f_{s+1}(x) = s+1$, $F_{s+1}(x) = 1$ and define a pending action $p_F(e, x) = 1$. Proceed to the next stage.

Assume that the requirement receiving our attention is $\mathcal{R}_{e,n}$. If there is no restraint associated with $\mathcal{R}_{e,n}$ then define it as $\psi^{F_s}_{n,s}(x)$ in order to preserve $\Psi^{F_s}_{n,s}(x)$. Put $F_{s+1}(x) =$

$G_{s+1}(x) = 1 - F_s(x)$ ($F_s(x)$ is the same as $G_s(x)$). Moreover, set $f_{s+1}(x) = g_{s+1}(x) = s+1$ and $h_{s+1}(e, n, x) = \Psi^{F_s}_{n,s}(x)$.

4.4 Verification

Lemma 2. *Each requirement receives attention only finitely often.*

Proof. We proceed by induction on the priority ordering. Fix a requirement $\mathcal{P}$ and assume that all requirements $< \mathcal{P}$ receive attention only finitely often. Let t be the least stage for which there are no pending actions, no requirement $< \mathcal{P}$ receives attention at stages $\geq t$ and some x is reserved for $\mathcal{P}$ at stage t.

Suppose $\mathcal{P} = \mathcal{F}_e$ (handling $\mathcal{G}_e$ is analogous). If $\mathcal{F}_e$ does not receive attention at stages $> t$ then it receives attention only finitely often all along. Suppose $\mathcal{F}_e$ receives attention at some stage $> t$ and let $u + 1$ be the least. Hence, $f_u(x) = \Phi^{g_u}_{e,u}(x) \downarrow$. Since our construction always picks up a fresh witness, we have $f_u(x) = 0$. Observe that for all $v \geq u$, $g_v \upharpoonright \varphi^{g_u}_{e,u}(x) = g \upharpoonright \varphi^{g_u}_{e,u}(x)$, where $\varphi^{g_u}_{e,u}(x)$ is the restraint defined at stage $u + 1$ to preserve $\Phi^{g_u}_{e,u}(x) \downarrow$. Hence, for all $v \geq u+1$, $\Phi^{g_v}_{e,v}(x) \downarrow = 0$. Moreover, we have $f_v(x) = u+2$, for $v \geq u + 2$. It means that $\mathcal{F}_e$ does not receive attention at stages $> u + 1$. Therefore, $\mathcal{F}_e$ receives attention only finitely often.

Let $\mathcal{P} = \mathcal{R}_{e,n}$. We show that $\mathcal{R}_{e,n}$ receives attention no more than twice with x being reserved for it. Suppose $\mathcal{R}_{e,n}$ has received attention twice after stage t: let $u+1$ be the first and $v + 1$ be the second such stage. Let us denote $m := \Psi^{F_u}_{n,u}(x)$ and $m' := \Psi^{F_v}_{n,v}(x)$. We have $\phi_{e,u}(x, m) = F_u(x)$ and $m \geq h_u(e, n, x) = 0$. Moreover, we have $\phi_{e,v}(x, m') = F_v(x)$ and $m' \geq h_v(e, n, x) = m$. Because x is a fresh witness, we have $F_u(x) = 0$ and thus $\phi_{e,u}(x, m) = 0$. x is enumerated into (F_s) at stage $u + 1$ and stays there through stages $u + 1, u + 2, \ldots, v$. Hence, $F_v(x) = 1$ and thus $\phi_{e,v}(x, m') = 1$. But then we cannot have $m = m'$ and hence $m' > m$. Recall, however, that we have defined a restraint $r := \psi^{F_u}_{n,u}(x)$ at stage $u + 1$ and thus prevented requirements $> \mathcal{R}_{e,n}$ from changing $F_u \upharpoonright r$ (higher priority requirements cannot do that because of the inductive assumption). At stage $v + 1$ we withdraw x from (F_s) and hence $F_u(x) = F_{v+1}(x) = 0$. Therefore, we have $F_u \upharpoonright r = F_{v+1} \upharpoonright r$. We show that for all $w \geq v + 1$: $F_u \upharpoonright r = F_w \upharpoonright r$ and $h_w(e, n, x) = m'$. We have already shown this for $w = v + 1$. Assume $F_u \upharpoonright r = F_w \upharpoonright r$ and $h_w(e, n, x) = m'$, for a fixed stage $w \geq v+1$. Then $\Psi^{F_w}_{n,w}(x) = \Psi^{F_u}_{n,u}(x) = m < m' = h_w(e, n, x)$, so $\mathcal{R}_{e,n}$ does not receive attention at stage $w + 1$. Hence, $F_u \upharpoonright r = F_{w+1} \upharpoonright r$ and $h_{w+1}(e, n, x) = m'$. This means that $\mathcal{R}_{e,n}$ does not receive attention at stages $> v+1$ and, hence, $\mathcal{R}_{e,n}$ receives attention only finitely often. □

The construction is recursive and therefore $(f_s), (g_s), (F_s), (G_s)$ are computable families of computable sets. However, by Lemma 2, those families are in fact computable approximations. To see this, combine Lemma 2 with the observation that each number x is either never reserved for any requirement whatsoever or is reserved only once and only for one requirement. Therefore, the following limits exist: $\lim_s f_s = f$, $\lim_s g_s = g$, $\lim_s F_s = F$, $\lim_s G_s = G$.

Lemma 3. *Each requirement is eventually satisfied.*

Proof. Let $\mathcal{P}$ be any requirement. By Lemma 2, all requirements $< \mathcal{P}$ receive attention only finitely often. Let t be the least stage for which there are no pending actions, no requirements $< \mathcal{P}$ receive attention at stages $\geq t$ and some x is reserved for $\mathcal{P}$. Observe that t is the first stage at which x is reserved for $\mathcal{P}$. Since our construction always picks up a fresh witness, we have $f_t(x) = g_t(x) = F_t(x) = G_t(x) = 0$.

Suppose $\mathcal{P} = \mathcal{F}_e$. We have two cases: either $\exists u > t\, \Phi^{g_u}_{e,u}(x) \downarrow= 0$ or not. Suppose the latter. It follows that $\mathcal{F}_e$ does not receive attention at stages $u > t$ and hence $f_u(x) = 0$ for $u > t$. Therefore, $f(x) = \lim_s f_s(x) = 0$. Moreover, observe that $\neg\exists u > t\, \Phi^{g_u}_{e,u}(x) \downarrow= 0$ implies that either $\Phi^g_e(x) \uparrow$ or $\Phi^g_e(x) > 0$. Hence, $f(x) \not\simeq \Phi^g_e(x)$ and thus $\mathcal{F}_e$ is satisfied.

Now, suppose $\exists u > t\, \Phi^{g_u}_{e,u}(x) \downarrow= 0$ and let u be the least such stage. It means that $\mathcal{F}_e$ receives attention at stage $u + 1$ and $0 = f_u(x) = \Phi^{g_u}_{e,u}(x)$. Observe that for all $v \geq u$, $g_v \restriction \varphi^{g_u}_{e,u}(x) = g \restriction \varphi^{g_u}_{e,u}(x)$, where $\varphi^{g_u}_{e,u}(x)$ is the restraint defined at stage $u + 1$ to preserve $\Phi^{g_u}_{e,u}(x)$. Hence, $\Phi^g_e(x) \downarrow= 0$. However, $\forall v > u + 1\, f_v(x) = u + 2$ so $f(x) = \lim_s f_s(x) = u + 2$. Therefore, $f(x) \neq \Phi^g_e(x)$.

Now, suppose $\mathcal{P} = \mathcal{R}_{e,n}$. To prove that $\mathcal{R}_{e,n}$ is eventually satisfied, assume Ψ^F_n is the least modulus of ϕ_e. We show that under this assumption, $\mathcal{R}_{e,n}$ does not receive attention at stages $> t$ or receives it only once and in both cases it becomes satisfied.

First, assume the contrary, that $\mathcal{R}_{e,n}$ receives attention twice or more at stages $> t$. In fact, we have shown (see Lemma 2) that $\mathcal{R}_{e,n}$ receives attention no more then twice while reserved for the same number. So, let $r + 1$, $r' + 1$ be the first and the second such stage, respectively. Hence, we have $\phi_{e,r}(x, \Psi^{F_r}_{n,r}(x)) = F_r(x)$, $\phi_{e,r'}(x, \Psi^{F_{r'}}_{n,r'}(x)) = F_{r'}(x)$ and $\Psi^{F_{r'}}_{n,r'}(x) \geq h(e, n, x, r') = \Psi^{F_r}_{n,r}(x)$. Recall, that we have defined a restraint $\psi^{F_r}_{n,r}(x)$ at stage $r + 1$, denote it by z, to protect $\Psi^{F_r}_{n,r}(x)$. We have $F_r(x) = 0$, $F_{r'+1}(x) = 0$ and $F_q(x) = 1$, for $r + 1 \leq q \leq r'$. No number $\leq z$ and other than x has been enumerated nor withdrawn by (F_s) at stages $r, r + 1, \ldots, r', r' + 1$. Hence, $F_r \restriction z = F_{r'+1} \restriction z$. Therefore, $\Psi^{F_{r'+1}}_{n,r'+1}(x) = \Psi^{F_r}_{n,r}(x)$. Let $v \geq r' + 1$. Since $\mathcal{R}_{e,n}$ does not receive attention at stages $> r' + 1$ and neither lower (by construction) nor higher priority requirements (by inductive assumption and our choice of t) can affect numbers $\leq z$, we have $F_v \restriction z = F_{r'+1} \restriction z$. Therefore $\Psi^{F_v}_{n,v}(x) = \Psi^{F_r}_{n,r}(x)$ and consequently $\Psi^F_n(x) = \Psi^{F_r}_{n,r}(x)$. Now, observe that we have $\phi_{e,r}(x, \Psi^{F_r}_{n,r}(x)) = F_r(x) = 0$, $\phi_{e,r'}(x, \Psi^{F_{r'}}_{n,r'}(x)) = F_{r'}(x) = 1$ and $\Psi^{F_{r'}}_{n,r'}(x) \geq \Psi^{F_r}_{n,r}(x)$. Hence, $\Psi^{F_{r'}}_{n,r'}(x) > \Psi^{F_r}_{n,r}(x)$. So ϕ_e changes its mind after stage $\Psi^{F_r}_{n,r}(x)$. Since $\Psi^F_n(x) = \Psi^{F_r}_{n,r}(x)$, Ψ^F_n is not the least modulus of ϕ_e, contrary to our assumption.

Second, suppose $\mathcal{R}_{e,n}$ never receives attention at stages $> t$. Hence, we have $F_w(x) = F_t(x) = 0$, for $w > t$, and thus $F(x) = 0$. Let u be the least stage $> t$ such that: $\Psi^{F_u}_{n,u}(x) \downarrow$, $\phi_{e,u}(x, \Psi^{F_u}_{n,u}(x)) \downarrow$ and $F_u \restriction \psi^{F_u}_{n,u}(x) = F \restriction \psi^{F_u}_{n,u}(x)$. We have $\Psi^{F_u}_{n,u}(x) = \Psi^F_n(x)$ and $\phi_{e,u}(x, \Psi^{F_u}_{n,u}(x)) = \phi_e(x, \Psi^F_n(x))$. Because $F_u(x) = 0$ and $\mathcal{R}_{e,n}$ does not receive attention at stage $u + 1$, we have $\phi_{e,u}(x, \Psi^{F_u}_{n,u}(x)) \downarrow\neq F_u(x) = 0$. Hence, $\phi_{e,u}(x, \Psi^{F_u}_{n,u}(x)) \downarrow= 1$. So $\phi_e(x, \Psi^F_n(x)) = 1$. Since Ψ^F_n is the least modulus of ϕ_e, we have $\lim_s \phi_e(x, s) = 1 \neq 0 = F(x)$ and hence $\mathcal{R}_{e,n}$ is satisfied.

Third, suppose $\mathcal{R}_{e,n}$ receives attention at stages $> t$ only once and let $r + 1$ be that stage. We have $\phi_{e,r}(x, \Psi^{F_r}_{n,r}(x)) = F_r(x) = 0$. Now, since $\mathcal{R}_{e,n}$ does not receive attention later on, we have $F_v(x) = 1$, for $v \geq r + 1$ and hence $F(x) = 1$. Let u be the least stage $\geq r + 1$ such that: $\Psi^{F_u}_{n,u}(x) \downarrow$, $\phi_{e,u}(x, \Psi^{F_u}_{n,u}(x)) \downarrow$ and $F_u \restriction \psi^{F_u}_{n,u}(x) = F \restriction \psi^{F_u}_{n,u}(x)$ which gives us $\Psi^{F_u}_{n,u}(x) = \Psi^F_n(x)$ and $\phi_{e,u}(x, \Psi^{F_u}_{n,u}(x)) = \phi_e(x, \Psi^F_n(x))$. Since $\mathcal{R}_{e,n}$ does not receive attention at stages $> r + 1$, we have either i) $\phi_{e,u}(x, \Psi^{F_u}_{n,u}(x)) \neq F_u(x)$ or ii) $\Psi^{F_u}_{n,u}(x) < h_u(e, n, x)$. Suppose i). $F_u(x) = 1$ so $\phi_{e,u}(x, \Psi^{F_u}_{n,u}(x)) = 0$. It means that $\phi_e(x, \Psi^F_n(x)) = 0$. Since Ψ^F_n is the least modulus of ϕ_e, we have $\lim_s \phi_e(x, s) = 0 \neq 1 = F(x)$ and thus $\mathcal{R}_{e,n}$ is satisfied. Now, suppose ii). We have $h_u(e, n, x) = h_{r+1}(e, n, x) = \Psi^{F_r}_{n,r}(x)$. So $\Psi^F_n(x) < \Psi^{F_r}_{n,r}(x)$. Since Ψ^F_n is the least modulus of ϕ_e and $\phi_e(x, \Psi^{F_r}_{n,r}(x)) = 0$, we have $\lim_s \phi_e(x, s) = 0 \neq 1 = F(x)$ and thus $\mathcal{R}_{e,n}$ is satisfied. □

Lemma 4. *f and g are least moduli for (F_s) and (G_s), respectively.*

Proof. We show f is the least modulus of (F_s) (the case of g and (G_s) is symmetrical). Recall that we start from $f_0(x) = F_0(x) = 0$ and observe that changes we made in our construction to (f_s) and (F_s) satisfy the following:

i) $f_t(x) \neq f_{t+1}(x) \Rightarrow f_{t+1}(x) = t + 1$, and

ii) $f_t(x) \neq f_{t+1}(x) \Leftrightarrow F_t(x) \neq F_{t+1}(x)$.

From this, it follows easily that $\lim_s f_s(x) = f(x)$ is precisely the first stage at which (F_s) settles down for argument x. □

Lemma 5. $\lim_s F_s = \lim_s G_s$.

Proof. When (F_s) and (G_s) are modified in an attempt to satisfy some $\mathcal{R}_{e,n}$, both approximations change in the same way. However, if they are modified in an attempt to satisfy $\mathcal{F}_e$ or $\mathcal{G}_e$, we make them differ at one stage but we immediately define a pending action which is handled at the next stage at which we undo the last change. □

5 Conclusions

Least moduli of a given set provide another way of examining c.e. degrees lying in its upper cone. And vice versa: results concerning structural properties of c.e. degrees above a given set are informative for detecting the behaviour of its least moduli. For example, we have observed that the non-density theorem for the 2-c.e. degrees implies that all least moduli of the 2-c.e. degree $< \underline{0}'$ constructed in the theorem are to be found in $\underline{0}'$. We have also shown how one may exploit the properties of least moduli to carry out the construction of a non-c.e. degree, which is somewhat different from Cooper's technique [6]. There are further questions which may be of some interest in view of Theorem 4. For example, we have not established whether the non-c.e. degree constructed in Theorem 5 is low or whether the join of its moduli is $\underline{0}'$. Nevertheless, it seems likely that the answers are positive.

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